Differential Geometry I: Curves and surfaces

5.1 **(a)** Let $\gamma : I \to \mathbb{R}^n$ be a regular C^2 curve. Prove that it is biregular if and only if $\kappa_\gamma(t) \neq 0$.

(b) Let $\gamma : I \to \mathbb{R}^3$ be a Frenet regular curve. Prove that its torsion is *geometric*, i.e. for any C^3 reparametrization $h : J \to I$, if $\tilde{\gamma} = \gamma \circ h$ then we have

$$\tau_\gamma(h(u)) = \tau_{\tilde{\gamma}}(u) \quad \text{for all } \ u \in J.$$

Solution. (a) If the curve γ is biregular, then $\dot{\gamma}$ and $\ddot{\gamma}$ are linearly independent (note that $\dot{\gamma} \neq 0$ since the curve is assumed to be regular). The curvature vector is given (in view of the acceleration formula) by the relation

$$K_\gamma = \frac{1}{V_\gamma^2}\left(\ddot{\gamma} - \dot{V}_\gamma T_\gamma\right).$$

Thus, K_γ cannot vanish, since that would imply that $\ddot{\gamma} = \dot{V}_\gamma T_\gamma // T_\gamma // \dot{\gamma}$, which would contradict the fact that $\dot{\gamma}$ and $\ddot{\gamma}$ are linearly independent. Thus, $K_\gamma \neq 0$ and, therefore, $\kappa_\gamma = \|K_\gamma\| \neq 0$.

Conversely, if $\kappa_\gamma \neq 0$ (and, therefore, $K_\gamma \neq 0$), we have (again by the acceleration formula) that

$$\ddot{\gamma} - \dot{V}_\gamma T_\gamma \neq 0.$$

Since $\dot{V}_\gamma = \langle \ddot{\gamma}, T_\gamma \rangle$ (as we have seen in class), the vector $\dot{V}_\gamma T_\gamma$ is the projection of $\ddot{\gamma}$ on T_γ. If $\ddot{\gamma}$ was parallel to T_γ, we would therefore have that $\ddot{\gamma} = \dot{V}_\gamma T_\gamma$, which would be a contradiction in view of the above relation. Therefore, $\ddot{\gamma}$ and $\dot{\gamma}$ are linearly independent, hence γ is biregular.

(b) Recall that, for a Frenet regular curve γ in $\mathbb{R}^3$, the torsion τ_γ is defined by

$$\tau_\gamma = \left\langle \frac{1}{V_\gamma}\frac{d}{dt}N_\gamma, B_\gamma \right\rangle,$$

where N_γ is the principal normal and $B_\gamma = T_\gamma \times N_\gamma$ is the binormal. Note that, under a reparametrization as in the statement of the exercise:

$$\frac{d}{du}\tilde{\gamma}(u) = \frac{d}{du}\left(\gamma(h(u))\right) = h'(u)\dot{\gamma}(h(u))$$

and

$$\frac{d^2}{du^2}\tilde{\gamma}(u) = \frac{d^2}{du^2}\left(\gamma(h(u))\right) = h'(u)\ddot{\gamma}(h(u)) + h''(u)\dot{\gamma}(h(u)).$$

We will consider two cases:

- In the case when h is a direct reparametrization, i.e. when $h' > 0$, we have

$$V_{\tilde{\gamma}}(u) = h'(u)V_\gamma(h(u))$$

  and

$$T_{\tilde{\gamma}}(u) = \frac{\frac{d}{du}\tilde{\gamma}(u)}{V_{\tilde{\gamma}}(u)} = \frac{h'(u)\dot{\gamma}(h(u))}{h'(u)V_\gamma(h(u))} = T_\gamma(h(u)).$$

Moreover, using the above formula for $\frac{d^2}{du^2}\tilde{\gamma}(u)$:

$$N_{\tilde{\gamma}}(u) = \frac{\frac{d^2}{du^2}\tilde{\gamma}(u) - \big\langle \frac{d^2}{du^2}\tilde{\gamma}(u), T_{\tilde{\gamma}}(u)\big\rangle T_{\tilde{\gamma}}(u)}{\big\| \frac{d^2}{du^2}\tilde{\gamma}(u) - \big\langle \frac{d^2}{du^2}\tilde{\gamma}(u), T_{\tilde{\gamma}}(u)\big\rangle T_{\tilde{\gamma}}(u)\big\|}$$

$$= \frac{h'(u)\ddot{\gamma}(h(u)) + h''(u)\dot{\gamma}(h(u)) - \big\langle h'(u)\ddot{\gamma}(h(u)) + h''(u)\dot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))}{\big\|h'(u)\ddot{\gamma}(h(u)) + h''(u)\dot{\gamma}(h(u)) - \big\langle h'(u)\ddot{\gamma}(h(u)) + h''(u)\dot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))\big\|}.$$

Note that, in the above expression, since $\dot{\gamma}(h(u))$ is parallel to $T_\gamma(h(u))$, we have that

$$h''(u)\dot{\gamma}(h(u)) - \big\langle h''(u)\dot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u)) = 0,$$

so the expression simplifies to

$$\begin{aligned} N_{\tilde{\gamma}}(u) &= \frac{h'(u)\ddot{\gamma}(h(u)) - \big\langle h'(u)\ddot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))}{\big\|h'(u)\ddot{\gamma}(h(u)) - \big\langle h'(u)\ddot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))\big\|} \\ &= \frac{h'(u)\Big(\ddot{\gamma}(h(u)) - \big\langle \ddot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))\Big)}{\Big\|h'(u)\Big(\ddot{\gamma}(h(u)) - \big\langle \ddot{\gamma}(h(u)), T_\gamma(h(u))\big\rangle T_\gamma(h(u))\Big)\Big\|} \\ &= N_\gamma(h(u)). \end{aligned}$$

As a result, we also have

$$B_{\tilde{\gamma}}(u) = T_{\tilde{\gamma}}(u) \times N_{\tilde{\gamma}}(u) = T_\gamma(h(u)) \times N_\gamma(h(u)) = B_\gamma(h(u)).$$

Lastly, recall that the derivative operator $\frac{1}{V_\gamma(t)}\frac{d}{dt}$ is invariant under direct reparametrizations, in the sense that, for any smooth function f,

$$\frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du}\left(f(h(u))\right) = \frac{1}{|h'(u)|V_\gamma(h(u))} f'(h(u))h'(u) = \frac{1}{V_\gamma(t)} f'(t) \quad \text{for } t = h(u).$$

Combining the above observations, we infer that

$$\begin{aligned} \tau_{\tilde{\gamma}}(u) &= \Big\langle \frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du} N_{\tilde{\gamma}}(u), B_{\tilde{\gamma}}(u)\Big\rangle \\ &= \Big\langle \frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du}\big(N_\gamma(h(u))\big), B_\gamma(h(u))\Big\rangle \\ &= \Big\langle \frac{1}{V_\gamma(h(u))}\dot{N}_\gamma(h(u)), B_\gamma(h(u))\Big\rangle \\ &= \tau_\gamma(h(u)). \end{aligned}$$

- In the case when h is an inversion, i.e. when $h' < 0$, we similarly

$$V_{\tilde{\gamma}}(u) = |h'(u)|V_\gamma(h(u)) = -h'(u)V_\gamma(h(u))$$

and

$$T_{\tilde{\gamma}}(u) = \frac{\frac{d}{du}\tilde{\gamma}(u)}{V_{\tilde{\gamma}}(u)} = \frac{h'(u)\dot{\gamma}(h(u))}{-h'(u)V_\gamma(h(u))} = -T_\gamma(h(u)).$$

Also, the formula for $\frac{d^2}{du^2}\tilde{\gamma}(u)$ implies (repeating the same calculations as before) that:

$$N_{\tilde{\gamma}}(u) = N_\gamma(h(u))$$

(note that, unlike T_γ, the principal normal N_γ does not change direction under an inversion). Thus,

$$B_{\tilde{\gamma}}(u) = T_{\tilde{\gamma}}(u) \times N_{\tilde{\gamma}}(u) = -T_\gamma(h(u)) \times N_\gamma(h(u)) = -B_\gamma(h(u)).$$

Lastly, the derivative operator $\frac{1}{V_\gamma(t)}\frac{d}{dt}$ under an inversion changes sign:

$$\frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du}\left(f(h(u))\right) = \frac{1}{|h'(u)|V_\gamma(h(u))}f'(h(u))h'(u) = -\frac{1}{V_\gamma(t)}f'(t) \quad \text{for } t = h(u).$$

Combining the above observations, we infer that

$$\begin{aligned}
\tau_{\tilde{\gamma}}(u) &= \Big\langle \frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du}N_{\tilde{\gamma}}(u), B_{\tilde{\gamma}}(u)\Big\rangle \\
&= \Big\langle \frac{1}{V_{\tilde{\gamma}}(u)}\frac{d}{du}\big(N_\gamma(h(u))\big), -B_\gamma(h(u))\Big\rangle \\
&= \Big\langle -\frac{1}{V_\gamma(h(u))}\dot{N}_\gamma(h(u)), -B_\gamma(h(u))\Big\rangle \\
&= \tau_\gamma(h(u)).
\end{aligned}$$

5.2 Prove that the curve $\gamma(t) = (\cosh t, \sinh t, t)$ is biregular of class C^3, and compute its curvature vector, curvature (the curvature is the norm of the curvature vector) and torsion.

Solution. The curve γ is biregular since

$$\dot{\gamma}(t) = (\sinh t, \cosh t, 1), \quad \ddot{\gamma}(t) = (\cosh t, \sinh t, 0)$$

are linearly independent for all t. It is also of class C^∞, since its coordinates are infinitely smooth functions.

The speed is $V_\gamma(t) = \sqrt{2}\cosh t$, hence

$$T_\gamma(t) = \frac{\dot{\gamma}(t)}{V_\gamma(t)} = \frac{1}{\sqrt{2}}(\tanh t, 1, \tfrac{1}{\cosh t}).$$

The curvature vector is

$$K_\gamma(t) = \frac{1}{V_\gamma(t)}\dot{T}_\gamma(t) = \frac{1}{2\cosh^3 t}(1, 0, -\sinh t),$$

and the curvature is $\kappa_\gamma(t) = \frac{1}{2\cosh^2 t}$. The principal normal is

$$N_\gamma(t) = \frac{1}{\kappa_\gamma(t)} K_\gamma(t) = \left(\tfrac{1}{\cosh(t)}, 0, -\tanh(t)\right)$$

and the binormal is

$$B_\gamma(t) = T_\gamma(t) \times N_\gamma(t) = \frac{1}{\sqrt{2}}\left(-\tanh(t), 1, -\tfrac{1}{\cosh(t)}\right).$$

Thus,

$$\begin{aligned}
\tau_\gamma(t) &= \left\langle \frac{1}{V_\gamma} \dot{N}_\gamma, B_\gamma \right\rangle \\
&= \frac{1}{\sqrt{2}\cosh(t)} \left\langle \left(-\frac{\sinh(t)}{\cosh^2(t)}, 0, \tanh^2(t) - 1\right), \frac{1}{\sqrt{2}}\left(-\tanh(t), 1, -\tfrac{1}{\cosh(t)}\right) \right\rangle \\
&= \frac{1}{2\cosh^2(t)}.
\end{aligned}$$

5.3 Consider the curve

$$\gamma(t) = (\cos t + t \sin t,\ \sin t - t\cos t,\ t^2), \quad t \in \mathbb{R}.$$

(a) Find the singular point(s) of this curve.

(b) Compute the arc length parameter $s = s(t)$ from the initial point $\gamma(0)$.

For the remaining questions, restrict to $t > 0$.

(c) Compute the tangent vector $T_\gamma(t)$ and the curvature vector $K_\gamma(t)$.

(d) Determine the biregular points of γ.

(e) Compute the curvature $\kappa_\gamma(t)$ and the principal normal vector $N_\gamma(t)$.

(f) Give the binormal vector $B_\gamma(t)$ (at biregular points).

(g) Compute the torsion of γ.

Solution. (a) $\dot{\gamma}(t) = t(\cos t, \sin t, 2)$, so $V_\gamma(t) = |t|\sqrt{5}$. The only singular point is $t = 0$.

(b) The arc length is

$$s(t) = \int_0^t \|\dot{\gamma}(u)\|\, du = \sqrt{5}\int_0^t |u|\, du = \operatorname{sgn}(t)\frac{\sqrt{5}}{2}t^2.$$

(c) For $t > 0$, $T_\gamma(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} = \frac{\sqrt{5}}{5}(\cos t, \sin t, 2)$, so $\dot{T}_\gamma(t) = \frac{\sqrt{5}}{5}(-\sin t, \cos t, 0)$ and

$$K_\gamma(t) = \frac{1}{V_\gamma(t)}\dot{T}_\gamma(t) = \frac{1}{5t}(-\sin t, \cos t, 0).$$

(d) The curve is biregular for all $t > 0$ since $\dot{\gamma}$ and $\ddot{\gamma}$ are linearly independent.
(e) $\kappa_\gamma(t) = \|K_\gamma(t)\| = \frac{1}{5t}$ and $N_\gamma(t) = (-\sin t, \cos t, 0)$.
(f) $B_\gamma(t) = T_\gamma(t) \times N_\gamma(t) = \frac{\sqrt{5}}{5}(-2\cos t, -2\sin t, 1)$.
(g) $\dot{N}_\gamma(t) = -(\cos t, \sin t, 0)$, hence

$$\tau_\gamma(t) = \frac{1}{V_\gamma(t)}\langle B_\gamma(t), \dot{N}_\gamma(t)\rangle = \frac{2}{5t}.$$

5.4 Let $\gamma : I \to \mathbb{R}^3$ be a Frenet regular curve. The *Darboux vector* of γ is the vector field along γ defined by

$$D_\gamma(u) := \tau_\gamma(u)T_\gamma(u) + \kappa_\gamma(u)B_\gamma(u).$$

Show that for any vector field A along γ written as $A(u) = a_1(u)T(u)+a_2(u)N(u)+a_3(u)B(u)$, we have

$$\frac{1}{V}\frac{dA}{du} = \frac{1}{V}(\dot{a}_1 T + \dot{a}_2 N + \dot{a}_3 B) + D \times A.$$

(This is the *Darboux formula.*)

Solution. Using the Serret–Frenet equations:

$$\frac{1}{V}\frac{dA}{du} = \frac{1}{V}(\dot{a}_1 T + \dot{a}_2 N + \dot{a}_3 B) + \frac{1}{V}(a_1\dot{T} + a_2\dot{N} + a_3\dot{B}).$$

But

$$D \times A = (\tau T + \kappa B) \times (a_1 T + a_2 N + a_3 B) = \frac{1}{V}(a_1\dot{T} + a_2\dot{N} + a_3\dot{B}),$$

giving the claimed formula.

Remark. This expresses the derivative of a vector in the moving Frenet frame as the sum of a relative derivative in the moving frame and a term describing the instantaneous rotation $D \times A$.
Example. For $A(t) = V_\gamma(t)T_\gamma(t)$, we get

$$\ddot{\gamma} = \dot{V}T + V^2 D \times T = \dot{V}T + V^2\kappa N.$$

5.5 Compute the Darboux vector of the right circular helix $\gamma(u) = (a\cos u,\, a\sin u,\, bu)$.

Solution. For this helix,

$$T = \frac{1}{c}(-a\sin u, a\cos u, b), \quad B = \frac{1}{c}(b\sin u, -b\cos u, a),$$

$$\kappa = \frac{a}{c^2}, \quad \tau = \frac{b}{c^2}, \quad c = \sqrt{a^2+b^2}.$$

Thus

$$D = \tau T + \kappa B = \left(0, 0, \frac{a^2+b^2}{c^3}\right),$$

a constant vector. The constant angle between D and T confirms the helix has constant slope.

5.6 Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (t,\, t^2 + |t|^3,\, 0).$$

Show that this curve is *regular in the Frenet sense*, but not of class C^3. Then compute its Frenet frame.

Solution. γ is C^2 but not C^3 since

$$\dot\gamma(t) = (1, 2t + 3t|t|, 0), \quad \ddot\gamma(t) = (0, 2 + 6|t|, 0).$$

These are continuous and linearly independent, so γ is biregular. Let $V(t) = \|\dot\gamma(t)\|$. The principal normal (by Gram–Schmidt) is

$$N(t) = \frac{1}{V(t)}(-2t - 3t|t|, 1, 0).$$

Since N is C^1, the curve is Frenet-regular. It lies in the plane Oxy, so $B = (0, 0, 1)$ is constant. The Frenet frame is:

$$T(t) = \frac{1}{V(t)}(1, 2t + 3t|t|, 0), \quad N(t) = \frac{1}{V(t)}(-2t - 3t|t|, 1, 0), \quad B(t) = (0, 0, 1).$$

Remark. Any planar biregular curve is Frenet-regular, since, in that case, B is constant (and equal to the unit normal to the plane) and so $N = B \times T$ is as regular as T.

5.7 What can be said about a (Frenet-regular) curve whose curvature and torsion are both constant?

Solution. Two curves with equal curvature and torsion (under natural parametrization) differ only by a rigid motion. The right circular helix has constant curvature and torsion, hence any Frenet-regular curve with constant curvature and torsion is a right circular helix. Degenerate cases: if torsion $= 0$, the curve is planar (a circle); if curvature $= 0$, the curve is a line.

5.8 Show that the torsion of a C^3 biregular curve $\gamma : I \to \mathbb{R}^3$ can be computed by the formula

$$\tau(u) = \frac{[\dot\gamma(u), \ddot\gamma(u), \dddot\gamma(u)]}{\|\dot\gamma(u) \times \ddot\gamma(u)\|^2} = \frac{[\dot\gamma, \ddot\gamma, \dddot\gamma]}{\kappa^2(u) V_\gamma^6(u)},$$

where $[x, y, z] = \langle x,\, y \times z\rangle$ denotes the mixed triple product of three vectors in $\mathbb{R}^3$.

Solution. Differentiating $\ddot\gamma = \dot V T + V^2 \kappa N$ and taking the dot product with B, the Serret–Frenet equations give

$$\langle \dddot\gamma, B\rangle = (V^3\kappa)\tau.$$

Since $\|\dot\gamma \times \ddot\gamma\| = V^3\kappa$, we obtain

$$\tau = \frac{\langle \dot\gamma \times \ddot\gamma,\, \dddot\gamma\rangle}{\|\dot\gamma \times \ddot\gamma\|^2}.$$

5.9 Show that a C^3 biregular curve $\gamma : I \to \mathbb{R}^3$ is a right circular helix if and only if its Darboux vector is constant.

Solution. From Exercise 5.5, the Darboux vector of a circular helix is constant. Conversely, since $D = \tau T + \kappa B$ and thus $\tau = \langle T, D\rangle$, differentiating this relation and using the assumption that D is constant, we get

$$\frac{1}{V}\frac{d\tau}{du} = \frac{1}{V}\langle \dot{T}, D\rangle = \kappa\langle N, D\rangle = \kappa\langle N, \tau T + \kappa B\rangle = 0,$$

so τ is constant. A similar argument (differentiating $\langle B, D\rangle$ gives constant κ. Thus γ is a circular helix.

B. Additional Exercise

5.10 It is known that, up to rigid motion, the geometry of a curve is completely determined by its curvature and torsion. Hence, any geometric property can be expressed as one or more equations involving τ and κ. The goal of this exercise is to illustrate this fact in the case of *spherical curves* (i.e. curves lying on a sphere).

(a) Let $\gamma : I \to \mathbb{R}^3$ be a C^2 and regular curve parametrized by arc length and satisfying $\|\gamma(s)\| = r =$ constant. Prove that γ is biregular.

(b) Let $\gamma : I \to \mathbb{R}^3$ be a C^3 regular curve parametrized by arc length, satisfying $\|\gamma(s)\| = r =$ constant and such that it has nonzero torsion. Show that for all s,

$$\gamma(s) + \rho(s)N(s) + \frac{\dot{\rho}(s)}{\tau(s)}B(s) = 0,$$

where $\kappa(s)$ is the curvature of γ and $\rho(s) = \frac{1}{\kappa(s)}$ is the radius of curvature. Deduce that the function

$$s \mapsto \rho(s)^2 + \left(\frac{\dot{\rho}(s)}{\tau(s)}\right)^2$$

is constant.

(c) Conversely, let $\gamma : I \to \mathbb{R}^3$ be a C^3 regular curve parametrized by arc length, with nonzero curvature and nonzero torsion. Show that γ is a spherical curve if and only if

$$\rho(s)^2 + \left(\frac{\dot{\rho}(s)}{\tau(s)}\right)^2$$

is constant. Determine the center and radius of the sphere.

Solution. (a) Since $\langle \gamma, \gamma\rangle =$ const, differentiating this relation we obtain

$$\langle \dot{\gamma}, \gamma\rangle = 0$$

(i.e. that $\dot{\gamma} \perp \gamma$) and, differenting once more,

$$\langle \ddot{\gamma}, \gamma \rangle + \langle \dot{\gamma}, \dot{\gamma} \rangle = 0 \Rightarrow \langle \ddot{\gamma}, \gamma \rangle = -\|\dot{\gamma}\|^2 < 0$$

(the above is a strict inequality, since $\dot{\gamma} \neq 0$). In particular, $\ddot{\gamma}$ is not perpendicular to γ. Therefore, $\dot{\gamma}$ and $\ddot{\gamma}$ are linearly independent (since, otherwise, $\ddot{\gamma}$ would have to be perpendicular to γ, like $\dot{\gamma}$). Thus, γ is biregular.

(b) From part (a), we have that γ is also biregular. Moreover, differentiating (like before) $\langle \gamma(s), \gamma(s) \rangle = \text{const}$ we have $\gamma \perp T$. Hence $\gamma(s) = a(s)N(s) + b(s)B(s)$ with $a^2 + b^2 = r^2$. Differentiating this relation (and using the Serret-Frenet formulas) gives

$$T = -a\kappa T + (a' - b\tau)N + (b' + a\tau)B,$$

so $a = -1/\kappa = -\rho$, $a' - b\tau = 0$, $b' + a\tau = 0$, hence $b = -\dot{\rho}/\tau$ and

$$\gamma = -\rho N - \frac{\dot{\rho}}{\tau} B.$$

Then $r^2 = \|\gamma\|^2 = \rho^2 + (\dot{\rho}/\tau)^2$ is constant.

(c) Conversely, define

$$c(s) = \gamma(s) + \rho(s)N(s) + \frac{\dot{\rho}(s)}{\tau(s)} B(s).$$

Differentiating and using the Serret–Frenet equations yields $\frac{d}{ds}c(s) = 0$, so c is constant. Thus $\|\gamma(s) - c\|^2 = \rho^2 + (\dot{\rho}/\tau)^2$ is constant; γ lies on the sphere with center c and radius

$$r = \sqrt{\rho^2 + \left(\frac{\dot{\rho}}{\tau}\right)^2}.$$